

## Discrete Structuren) Hertentamen II, 29 October 2007

The problems are to be solved within 3 hrs.

**The use of supporting material (books, notes, calculators) is not allowed.**

In each problem you can obtain 10 points, i.e. 100 in total. Your partial result for the first 5 problems may be replaced by your grade in the midterm exam ( $\times 5$ ), provided the grade was  $\geq 5.5$ .

**Some useful hints:**

- Give precise arguments for all your answers.
  - You can write in English or Dutch, but in any case use a readable font!
  - Counterexamples prove that a statement is not true, but positive examples do not prove general validity.
  - If you refer to the hand-out sheet, numbers of implications etc. are sufficient.
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1. Prove that the following proposition

$$((p \rightarrow q) \rightarrow q) \leftrightarrow (p \vee q)$$

is a tautology. Use the form of an *annotated linear proof* (geannoteerd lineair bewijs).

2. Prove (by cases) that  $|x + y| \leq |x| + |y|$  for  $x, y \in \mathbb{R}$ .

3. Prove by (infinite) mathematical induction:  $\sum_{i=0}^n (2i + 1) = (n + 1)^2$  for  $n \in \mathbb{N}$ .

4. Give an explicit expression for the sequence  $s_n$ , defined by

$$\begin{aligned} s_0 &= -1 \\ s_1 &= 4 \\ s_n &= 4s_{n-1} - 4s_{n-2} \quad \text{for } n \geq 2 \end{aligned}$$

5.

- (a) Let  $s(n)$  ( $n \in \mathbb{N}$ ) be a sequence. Define the meaning of  $s(n) = O(n)$  and of  $s(n) = \Theta(n)$ .  
(b) Are the following statements true or false? (Give precise arguments!)

$$2^{2^n} = O(2^n) \quad 2^{n+1} = \Theta(2^n)$$

6. Let the relation  $R$  on  $\mathbb{Z}$  be defined as:

$$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid (x^2 - 2xy + y^2) \text{ is even}\}.$$

Show explicitly that  $R$  satisfies the properties of an equivalence relation. What are the equivalence classes of  $R$ ?

7.

(a) Show that the proposition

$$[\exists x p(x)] \wedge [\exists x q(x)] \rightarrow \exists x [p(x) \wedge q(x)]$$

is not a tautology. You can do this by giving examples for  $p(x)$  and  $q(x)$  for which the proposition is false.

(b) Show that the proposition

$$\exists x \forall y p(x, y) \rightarrow \forall x \exists y p(x, y)$$

is not a tautology. Again it is sufficient to show that the proposition is false for a particular  $p(x, y)$ .

8. Let  $A$  be the Boolean matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Calculate  $A * A$ .

(b) Is the relation corresponding to  $A$  transitive? Explain your answer!

(c) Which matrices represent the symmetric closure, the reflective closure, and the transitive closure of the relation corresponding to  $A$ ?

9. After having graduated you have been hired by a manufacturer of computer hardware. Your first task is to specify a scheme for the serial number of a new product. You decide on using alphanumerical characters, i.e. the 26 capital letters and the 10 digits.

Your company does not expect to manufacture more than 1000000000 (i.e.  $10^9$ ) of these devices. Out of how many alphanumerical characters should the serial number consist, such that there will be a unique serial number for each manufactured device and the serial number of each device is as short as possible?

You should not give the result as a number; it is sufficient to provide an analytic expression, e.g.  $\exp[12]$  instead of 162754.7914....

10.

(a) How many edges are there in a complete graph with  $n = 10$  vertices?

(b) How many edges are there in a binary rooted tree with  $n = 43$  vertices?

(c) How many edges are there in a ternary rooted tree with  $n = 43$  vertices?